Limiting Relation for the Eddy Diffusivity Close to a Wall

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Experiments on mass transfer for fully developed flow in a pipe show that the eddy diffusivity for mass close to the wall is described by the functionality

$$\frac{\epsilon}{\nu} = g(y^+)$$

The best interpretation of presently available data in a pipe and in an annulus is given by the following, more explicit relation

$$\frac{\epsilon}{\nu} = 0.00032y^{+4}$$

The evidence for this latter relation is not as conclusive as one would like. Accurate measurements of concentration profiles close to a wall are needed.

There has been considerable discussion regarding the limiting law governing turbulent transport of heat and mass at small distances from a wall. Experiments at large Schmidt numbers offer an opportunity to settle this question. The results of studies on fully developed turbulent flows which have produced precise and consistent results are reviewed, and it is shown that for flow in a pipe or in an annulus that presently available experimental results indicate that as $y^+ \to 0$

$$\frac{\epsilon}{\nu} = 0.00032y^{+4} \tag{1}$$

This result is derived from measurements of the effects of the Reynolds number, of the length of the mass transfer section, and of the Schmidt number on the rate of mass transfer.

Meyerink and Friedlander (*13*) and Linton and Sherwood (*12*) studied dissolution of the wall of a pipe. Lin et al. (*10*) used diffusion-controlled electrode reactions to determine the mass transfer rate to the inner wall of an annulus. Lin et al. (*11*) have used interferometric techniques to measure concentration profiles for fully developed mass transfer to one of the walls of a rectangular section, ¾ in. by 1¼ in. Shaw and Hanratty (*17*) used the same techniques as did Lin et al. (*10*) to study fully developed mass transfer in a pipe. Shaw et al. (*18*) and Schütz (*16*) examined the effect of the length of the mass transfer section on the rate of mass transfer.

Upon examining the results of Lin et al. (*10*) we found that in some of their runs the mass transfer rate had not reached the limiting value attained for long mass transfer sections. Therefore, in correlating fully developed mass transfer results only part of their data could be used. To have consistent values of the Schmidt number the diffusion coefficients were calculated from data under conditions of laminar flow for all of the systems that were studied rather than for part of them as was done by Lin et al. (*10*).

Shaw et al. reported an inconsistency between their laminar mass transfer measurements and their turbulent mass transfer measurements to short transfer sections. This inconsistency has been found to be partially due to an error in the calibration of some of the orifices plates. As a result of this error the fully developed value of $K_\infty^+ = 3.01 \times 10^{-4}$ at a Schmidt number of 2,400 must be changed to $K_\infty^+ = 3.52 \times 10^{-4}$. This is in close agreement with the results of Schütz on the same system. All the corrected data of Shaw et al. have been checked in recent experiments by Son (*20*) and agreement is accomplished.

TURBULENT MASS TRANSFER AT LARGE SCHMIDT NUMBERS

When considering the transport of mass between a wall and a turbulently flowing fluid, the concentration gradients are usually much larger in a direction perpendicular to the wall (y coordinate) than in the direction of flow (x coordinate). It is convenient to characterize the effect of turbulence on the rate of transport in terms of an eddy diffusion coefficient ϵ defined as

$$J = (D + \epsilon)\frac{\partial C}{\partial y} \tag{2}$$

For small values of the molecular diffusion coefficient D (large Schmidt number) the concentration changes in the fluid occur in a region close to the wall where the velocity profile is linearly dependent on y

$$u = \frac{\tau_w}{\mu} y \tag{3}$$

Since the concentration boundary layer is so thin, it is reasonable to assume that ϵ is dependent on wall parameters at high Schmidt numbers so that

$$\frac{\epsilon}{\nu} = f(y^+, x^+) \tag{4}$$

Table 1. Fully Developed Local Mass Transfer Coefficients

N_{Re}	Electrode Diam. = 0.0159 in. $K^+_\infty \times 10^4$	Diam. = 0.0259 in. $K^+_\infty \times 10^4$	Diam. = 0.064 in. $K^+_\infty \times 10^4$	Diam. = 0.125 in. $K^+_\infty \times 10^4$
50,200	3.51	3.50		3.57
42,200		3.60	3.57	
37,700	3.55	3.53		
36,900		3.56		
28,200				3.50
23,200	3.53	3.45	3.40	
23,100		3.52		
22,100				3.52
20,600				3.55
18,600	3.51	3.46	3.42	
17,000				3.47
14,500		3.52		
12,800		3.58		
11,200	3.58	3.65	3.50	
11,000				3.56
8,700		3.44		
7,750	3.55	3.52	3.50	

Support for such a relation for large Schmidt numbers can be obtained from the research of Shaw et al. (*18*) and of Shaw and Hanratty (*17*) which are described in this paper. For the case of heat transfer to air Johnk and Hanratty (*7*) have shown that, owing to the time dependency of turbulent diffusion, ϵ can be dependent on x even though the turbulence properties of the field are not changing in the x direction. If such an effect exists at large Schmidt numbers, it would be confined to small values of x^+, close to the beginning of the concentration boundary layer. Therefore it seems reasonable to neglect the effect of time dependence and to assume that the eddy diffusivity depends only on y^+

$$\frac{\epsilon}{\nu} = g(y^+) \tag{5}$$

In the neighborhood of $y^+ = 0$ the function describing ϵ/ν can be expanded in a Taylor series. If the concentration boundary layer is thin enough, then only the leading term in the series need be used so that

$$\frac{\epsilon}{\nu} = b(N_{Sc})\, y^{+\,n} \tag{6}$$

In general the coefficient b could be a function of the Schmidt number. Arguments can be presented to show the exponent n should be three or greater (*4, 22*). Values of n of three (*5, 11*), four (*3*), and five (*15*) have been used.

Making use of Equations (2) and (3) and assuming that diffusion in the direction of flow is negligible, we obtain the following mass balance on the diffusing species since the concentration boundary layer is thin enough so that the curvature of the wall can be ignored.

$$y^+ \frac{\partial C}{\partial x^+} = \frac{\partial}{\partial y^+}\left[\left((N_{Sc})^{-1} + \frac{\epsilon}{\nu}\right)\frac{\partial C}{\partial y^+}\right] \tag{7}$$

If the concentration is normalized with respect to the bulk concentration, the boundary conditions are

$$\begin{aligned} C(x^+ > 0,\ 0) &= 0 \\ C(x^+, y^+ \to \infty) &= 1 \\ C(x^+ < 0,\ y^+) &= 1 \end{aligned} \tag{8}$$

Equations (7) and (8) can be solved for small values of x^+, since for this limiting case the concentration boundary layer is thin enough so that ϵ may be neglected. The integration is effected by combination of variables (8) and the result is

$$C = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \int_0^\eta e^{-\eta^3}\, d\eta \tag{9}$$

where

$$\eta = y^+ \,(N_{Sc}/9x^+)^{1/3} \tag{10}$$

Values of the integral in Equation (9) have been tabulated by Abramowitz (*1*). The local value of the dimensionless mass transfer coefficient calculated from Equation (9) is

$$K^+ = 0.54\,(x^+)^{-1/3}\,(N_{Sc})^{-2/3} \tag{11}$$

The mass transfer coefficient based on the average mass transfer rate over length L is calculated from Equation (11) to be

$$\langle K^+ \rangle = 0.81\,(L^+)^{-1/3}\,(N_{Sc})^{-2/3} \tag{12}$$

For very large values of x^+ the concentration profile becomes fully developed and the concentration gradient in the x direction over the cross section of the pipe is constant. A mass balance on a disk of fluid with a length dx and a radius equal to that of the pipe gives the concentration gradient at large x^+ as

$$-\frac{dC}{dx^+} = \frac{4K_\infty{}^+}{N_{Re}} \tag{13}$$

Since $K_\infty{}^+$ is of the order of 10^{-4}, while N_{Re} is of order 10^3, this term is quite small at large x^+. Equation (7) may be integrated, taking account of the smallness of (dC/dx^+) and using the relation for ϵ/ν given by Equation (6). The result is

$$C = \frac{\pi \int_0^{y^+} \dfrac{dy^+}{(N_{Sc})^{-1} + b(y^+)^n}}{nb^{1/n} \sin\left(\dfrac{\pi}{n}\right)\,(N_{Sc})^{-\left(\frac{n-1}{n}\right)}} \tag{14}$$

$$K_\infty{}^+ = \frac{n}{\pi} b^{1/n} \sin\left(\frac{\pi}{n}\right)\,(N_{Sc})^{-\left(\frac{n-1}{n}\right)} \tag{15}$$

RESULTS ON FULLY DEVELOPED MASS TRANSFER

Shaw and Hanratty (*17*) measured the mass transfer rate under fully developed conditions at a Schmidt number of 2,400 and over a range of Reynolds numbers of 8,000 to 50,000. It was found, as predicted by Equation (5), that $K_\infty{}^+$ is independent of Reynolds number. Twenty-nine measurements were made and the maximum deviation from the average was 4%. The average of these when corrected for an error that was made in the orifice

Table 2. Data of Meyerink and Friedlander for Fully Developed Mass Transfer from the Wall of a Tube

System	Schmidt No.	No. of runs	Average $K^+_\infty \times 10^4$	Standard deviation, %	N_{Re}
Benzoic acid–water	900	21	7.38	9	4,720 to 22,900
Cinnamic acid–water	900	16	7.31	7	8,800 to 22,600
Aspirin–water	850	14	8.73	6	4,800 to 25,500

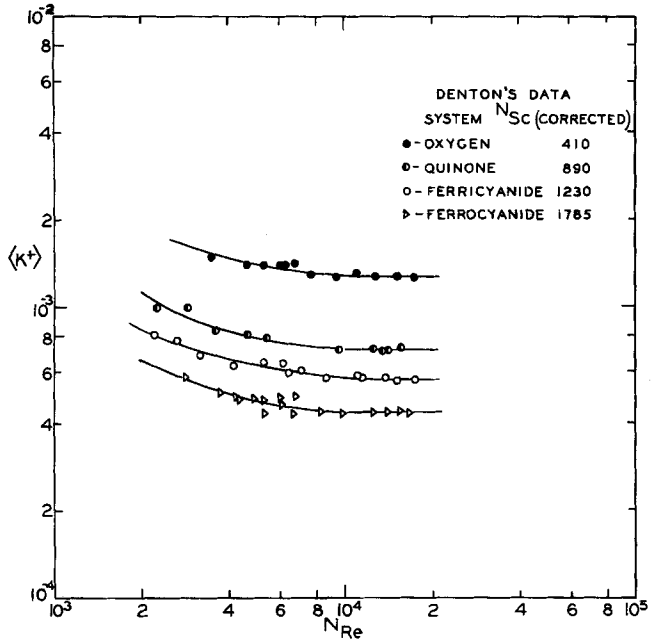

Fig. 1. Average mass transfer coefficient vs. Reynolds number.

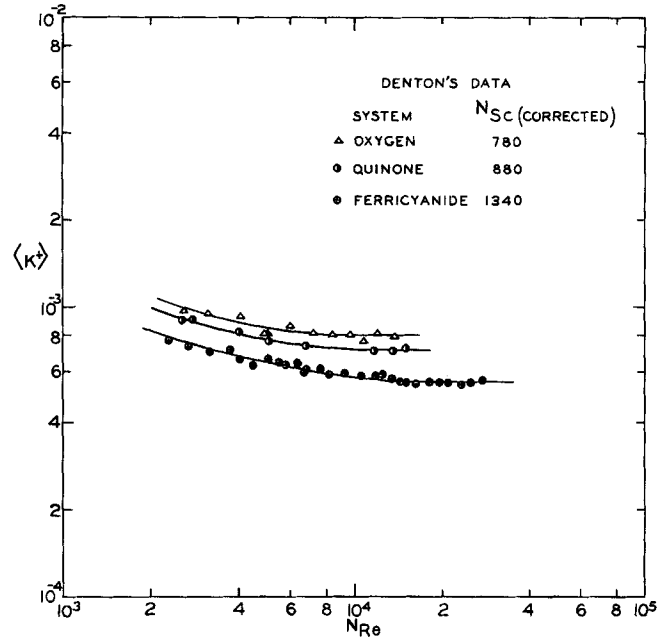

Fig. 2. Average mass transfer coefficient vs. Reynolds number.

calibration is 3.52×10^{-4}. The results of these experiments are summarized in Table 1.

Meyerink and Friedlander (*13*) investigated three different systems over the Reynolds number range 4,720 to 25,500; their results are summarized in Table 2. It is seen that excellent agreement is obtained between the benzoic acid–water system and the cinnamic acid–water system. However, the results with aspirin–water seem somewhat higher. The reason for this is not known.

The results of Linton and Sherwood on the dissolution of tubes showed too much scatter to calculate reliable values of $K_\infty{}^+$. It is quite possible that this scatter can be explained by the method used to fabricate the tubes. The test cylinders used by Meyerink and Friedlander were fabricated by pressing powders in a mold. Linton and Sherwood formed their tubes by melting the material and casting it in a mold. They reported that for benzoic acid small fissures and surface roughnesses developed while the dissolution process was occurring. However, such irregularities were not visible to the eye after dissolving their tubes made from cinnamic acid and beta-naphthol.

By using a number of electrochemical reactions Lin et al. (*10*) were able to carry out their studies of mass transfer to the inner wall of an annulus over a range of Schmidt numbers. Their measured mass transfer rates for the different reactions that they investigated are plotted in Figures 1 and 2. The dimensionless mass transfer rate is based on the friction velocity calculated from the shear stress on the inner wall of the annulus. The friction factor used for this calculation is given by Knudsen and Katz (*8*).

$$f = 0.055\ (N_{Re})^{-0.2} \left[\frac{d_2/d_1 - 1}{d_2/d_1} \right]^{0.1} \tag{16}$$

The friction velocity for the inner wall is

$$(u_i^*)^2 = \left[\frac{d_o + d_i}{d_o \dfrac{\tau_o}{\tau_i} + d_i} \right] \frac{f}{2} V^2 \tag{17}$$

Recent measurements (*2*) have shown that

$$\frac{\tau_i}{\tau_o} = 1.106 \text{ for } \frac{d_o}{d_i} = 1.97 \tag{18}$$

and this value of τ_i/τ_o has been substituted into Equation (17) to calculate the friction velocity u_i^*.

When plotted as in Figures 1 and 2, mass transfer measurements at large Schmidt numbers should be independent of Reynolds number if the mass transfer is fully developed. For each of the reactions used by Lin et al. (*10*), the value of K_∞ was taken as the limiting value at large Reynolds number. Since the length of the transfer section was constant, increasing N_{Re} corresponded to increasing L^+. Those runs for which a limiting value could not be determined were not used to evaluate K_∞.

For laminar flow in an annulus with a radius ratio of 1.97 the mass transfer rate to the inner wall is calculated as

$$\frac{\langle K \rangle d_i}{D} = 1.50 \left(\frac{W}{\rho D L} \right)^{1/3} \tag{19}$$

The mass transfer data for laminar flow, along with Equa-

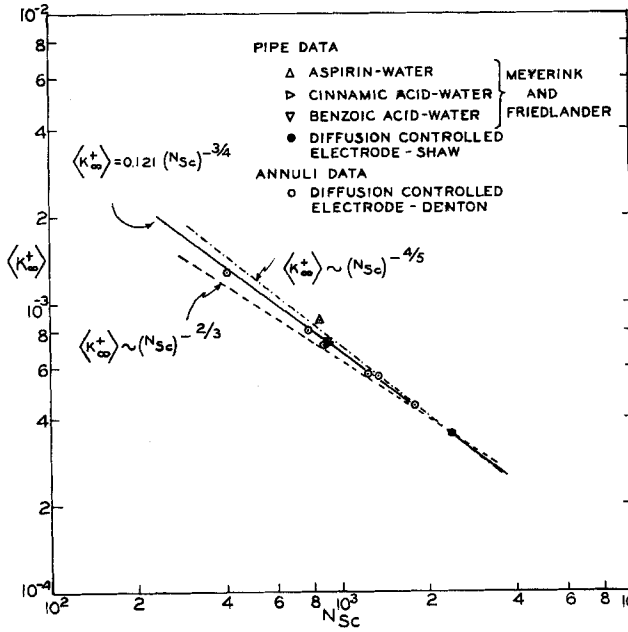

Fig. 3. Comparison of experimental K_∞^+ for high N_{Sc}.

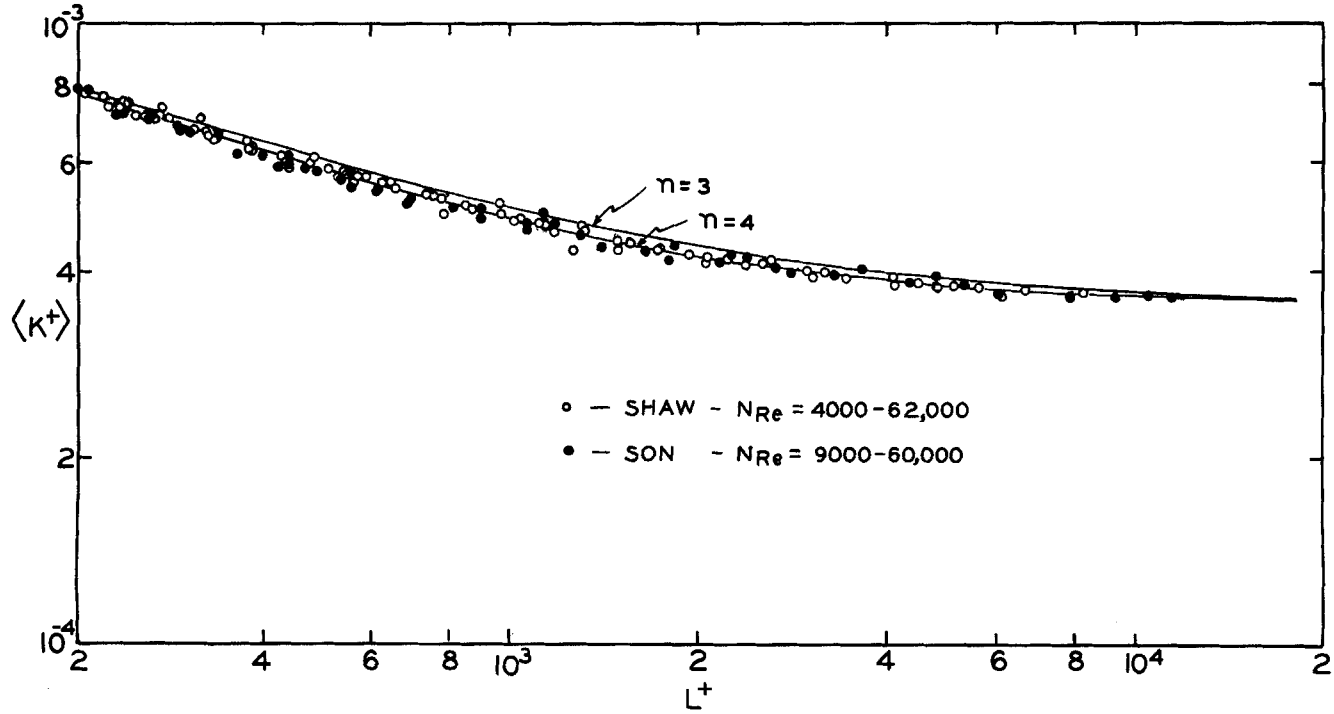

Fig. 4. Comparison of experimental K^+ with present analysis.

tion (19), were used to calculate the diffusion coefficients and therefore the Schmidt numbers corresponding to the different systems studied by Lin et al. (*10*). Although some error could be involved in using Equation (19), since the flow at low Reynolds numbers might not have corresponded exactly to the theoretical model, the relative magnitudes of the diffusion coefficients evaluated in this manner should be correct.

The values of K_∞^+ calculated from the data of Meyerink and Friedlander (*13*), the corrected data of Shaw and Hanratty (*17*), and the data of Lin et al. (*10*) are presented in Figure 3. All the results with the exception of those of Meyerink and Friedlander on aspirin–water can be correlated with the relation

$$K_\infty^+ = 0.121\ (N_{Sc})^{-3/4} \qquad (20)$$

For comparison the relations $K_\infty^+ \sim (N_{Sc})^{-2/3}$ and $K_\infty^+ \sim (N_{Sc})^{-4/5}$ are drawn in Figure 3. It is seen that these do not represent the results as well as Equation (20). By comparing Equation (20) with Equation (15) and by assuming that the coefficient b is not a function of Schmidt number, Equation (1) is obtained.

EFFECT OF THE LENGTH OF THE TRANSFER SECTION

Measurements by Son (*20*) and by Shaw et al. (*18*) on the effect of the length of the transfer section on the rate of mass transfer will now be examined. In these studies the length varied from 0.0177 to 4.31 pipe diameters and the Reynolds number varied from 4,000 to 62,000. The pipe diameter was 1 in. The results are plotted in Figure 4, where they are compared with calculated curves based on Equation (6) with $n = 3$ and $n = 4$. The calculations were carried out by integrating Equation (7) numerically with the use of the techniques described in reference *20*. The complete calculated values of K^+ and $\langle K^+ \rangle$ are shown in Figure 5.

The Schmidt number of 2,400 used in these calculations was determined by comparing Equation (12) with measurements in the laminar flow regime as shown in Figure 6. This Schmidt number is in good agreement with the value 2,440 calculated from the correlation obtained by Gordon and Tobias (*6*). The results for laminar flow include lengths ranging from 0.0174 to 1.91 pipe diameters and Reynolds numbers from 335 to 2,200. Data for turbulent flow for small L^+ for N_{Re} = 8,770 to 55,100 are compared with Equation (12) with the use of a Schmidt number of 2,400 in Figure 7. In both Figures 6 and 7 the data with the smallest transfer section showed the poorest agreement with Equation (12).

The coefficient b in Equation (6) was calculated from Equation (15) with the measured value of $K_\infty^+ = 3.52 \times 10^{-4}$. If a different value of b is used, then poor agreement with measurements at large L^+ is obtained. The values of b have been deliberately selected so that the calculated curves for all values of n coincide at large L^+. This is why all the values of n give a reasonable fit with the experimental results.

Even though the calculated relations between $\langle K^+ \rangle$ and L^+ do not appear to be too sensitive to the value of the exponent n, it is believed that the difference between the experiments and the calculations based on $n = 3$ is significant. It would appear that the results of these experiments would not support $n = 3$.

FULLY DEVELOPED CONCENTRATION PROFILES

Measurements of the concentration profile for fully developed mass transfer seem to offer the best opportunity to determine the limiting eddy diffusivity relation. Calculated fully developed concentration profiles for different values of n are shown in Figure 8.

An important study of the concentration field for mass transfer at large Schmidt numbers was done in a rectangular channel by Lin et al. (*11*). Because of the secondary flow which exists for turbulent flow in such a channel, it is not certain that the relations describing the variation of the eddy diffusion coefficient should be the same as for a pipe or an annulus. Furthermore, a comparison of the laminar flow data of Lin et al. (*11*) at N_{Re} = 2,000 (tests 71, 83, and 84) with the analytical solution [Equation (9)] based on an effective constant molecular diffusion coefficient shows discrepancies between theory and experiment as high as 12% (see Figure 9). Unless this

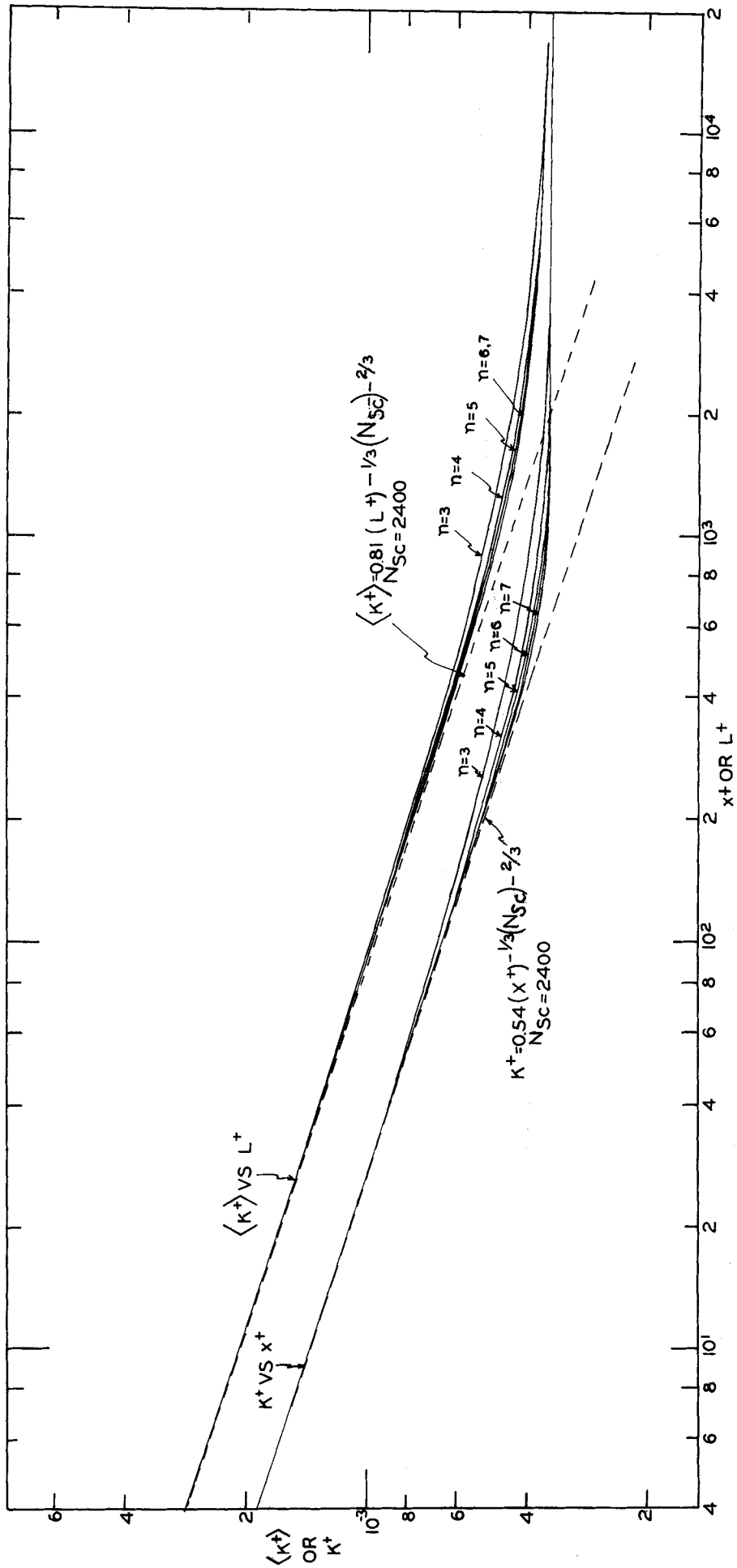

Fig. 5. Effect of the form of eddy diffusivity relations on the variation of local or average dimensionless mass transfer coefficient with x^+ or L^+.

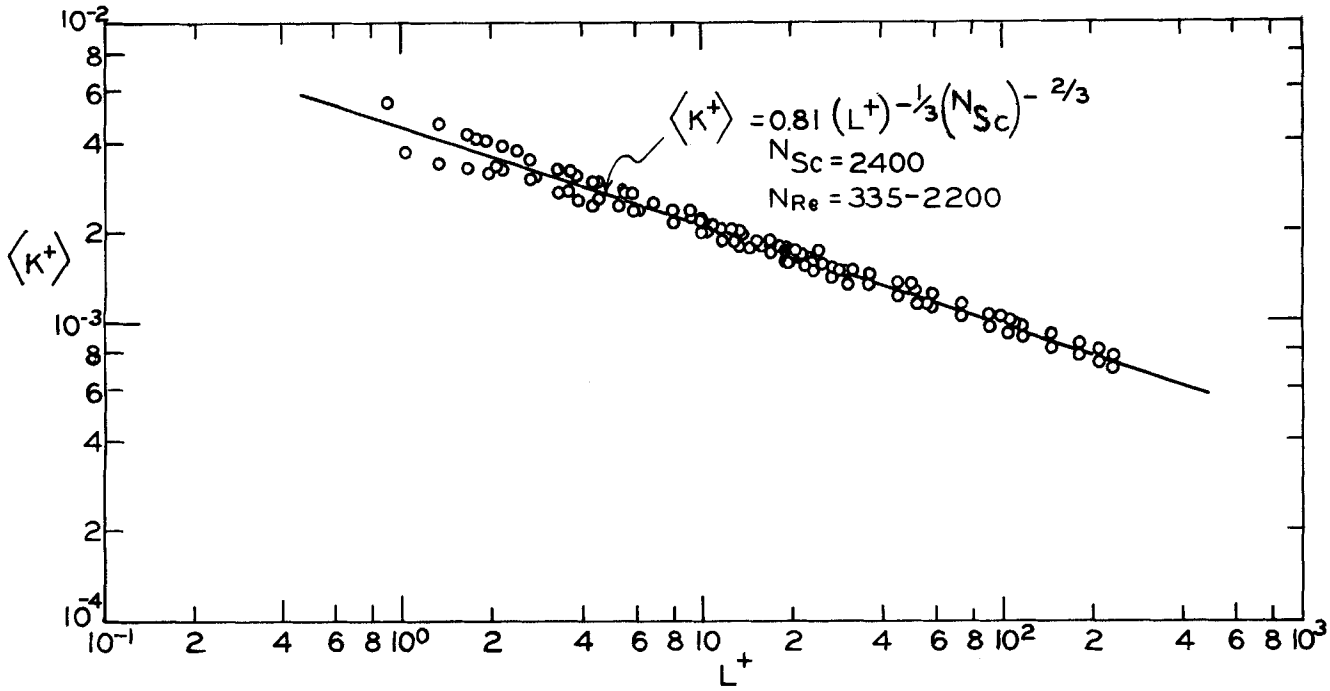

Fig. 6. Average mass transfer data, laminar flow.

matter is understood it does not seem that the concentration measurements presented by Lin et al. (*11*) for turbulent flow can be used to determine the proper value for n in the system used by them.

EFFECT OF TRUNCATION

The use of Equation (6) depends on the assumption that the concentration boundary layer is thin enough so that one need use only the first term in a Taylor series to describe the variation of the eddy diffusion coefficient with distance from the wall. The calculated mass transfer coefficients also depend on the assumption of a thin concentration boundary layer since Equation (3) was used to describe the velocity field. Calculated concentration profiles for Schmidt numbers of 2,400 and of 100 are shown in Figure 8. Since Equation (3) is valid for $y^+ < 5$, it is found that even at Schmidt numbers as low as 100 it is valid to assume a linear variation of the velocity over the concentration boundary layer.

To explore the truncation error involved in the use of Equation (6) relations of the form developed by Spalding (*19*) have been used:

$$\frac{\epsilon}{\nu} = A\left\{e^{By^+} - 1 - By^+ - \frac{(By^+)^2}{2} - \frac{(By^+)^3}{6}\right\} \quad (21)$$

$$\frac{\epsilon}{\nu} = A\left\{e^{By^+} - 1 - By^+ - \frac{(By^+)^2}{2}\right\} \quad (22)$$

Calculations based on these relations have indicated that truncation errors had a negligible effect on any of the calculated results presented in this paper and that they do not affect any of the conclusions reached in this paper. However, these arguments are not so forceful as one would like since the correctness of Equations (21) and (22) is questionable.

INTERPRETATION OF THE LIMITING RELATION

The limiting relation, Equation (1), which is supported by available experimental results, can be given an interpretation similar to that presented by Landau and Lifshitz (*9*). The statistical treatment of turbulent diffusion developed by Taylor (*21*) will be used for that purpose. If the scale of the motion responsible for turbulent diffusion is small compared with the characteristic length of the viscous sublayer (ν/u^*) and compared to the distance over which the root-mean-square turbulent velocity component perpendicular to a wall is varying appreciably, turbulent diffusion of particles can be assumed indepen-

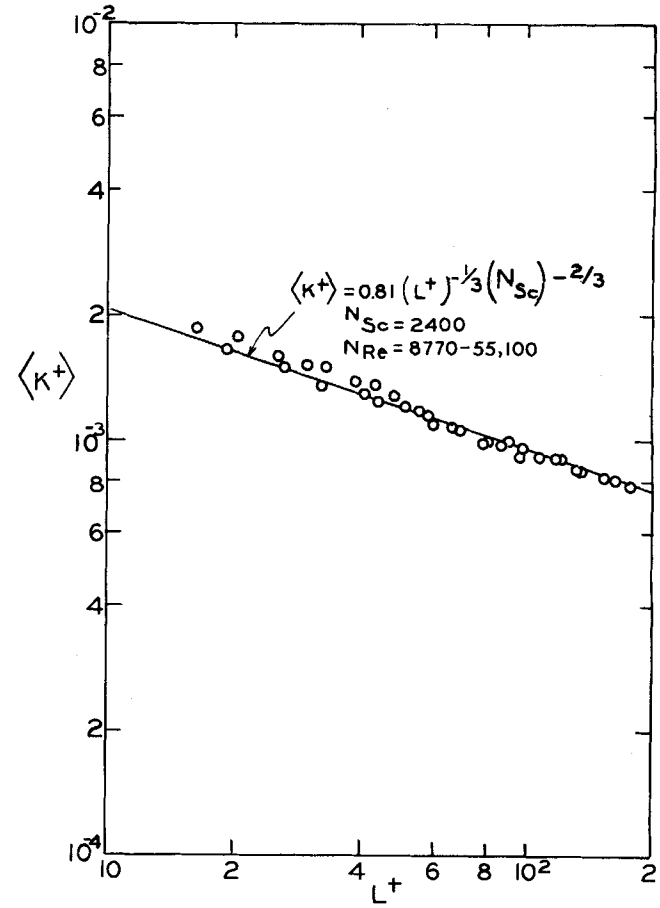

Fig. 7. Average mass transfer data, turbulent flow.

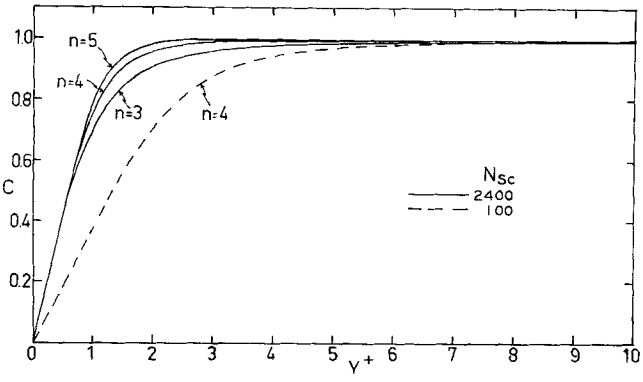

Fig. 8. Fully developed concentration profiles.

dent of their previous history. The turbulent diffusion coefficient is defined by Taylor (*21*) as

$$\epsilon = \overline{v^2}\ \tau \tag{23}$$

where τ is the Lagrangian time scale. From consideration of the continuity equation, it can be shown (*22*) for small distances from the wall that $\overline{v^2} \sim y^4$. If it is assumed that this relation is governed by wall parameters, then

$$\frac{(\overline{v^2})^{1/2}}{(u^*)} = E(y^+)^2 \tag{24}$$

and

$$\frac{\epsilon}{\nu} = E^2\left(\frac{\tau u^{*2}}{\nu}\right)(y^+)^4 \tag{25}$$

The measurements of mass transfer fluctuations by Shaw and Hanratty (*17*) indicate that $(\tau u^{*2}/\nu)$ should be approximately independent of the Reynolds number. If it is assumed further that τ is independent of y^+ for $y^+ \to 0$, then Equation (1) is obtained. Arguments in support of this latter assumption can be given similar to those presented by Landau and Lifshitz that "... it follows from the linearity of the equations of motion in the viscous sublayer ... that the periods of the turbulent eddies are the same throughout the thickness of the sublayer." From the measurement of the frequency spectrum of the mass transfer fluctuations presented by Shaw and Hanratty, it is possible to estimate the Eulerian time scale of the mass transfer fluctuations at the wall. If this can be used to estimate τ then one obtains

$$\left(\frac{\tau u^{*2}}{\nu}\right) \sim 5 \times 10^2 \tag{26}$$

From Equations (1) and (25) it is found that

$$E \sim 0.8 \times 10^{-3}$$

The ratio of the length scale for turbulent diffusion $[(\overline{v^2})^{1/2}\tau]$ to the characteristic length of the viscous sublayer is thus estimated as

$$\frac{(\overline{v^2})^{1/2}\tau}{(\nu/u^*)} \sim 0.4(y^+)^2 \tag{27}$$

This ratio is not as small as one would like for the above theoretical approach to be justified. However, owing to the roughness of the estimates, it is not large enough to discredit it.

The studies of Shaw and Hanratty (*17*) show that mass transfer to a solid surface is chaotic and that its rate has wide variations in magnitude. In fact, for a Schmidt number of 2,400, the ratio of the root-mean-square value of the fluctuations in the mass transfer coefficient to the time-averaged value equals 0.48. This would suggest that a mechanism of the diffusion type such as is outlined above might not be the best one for explaining the role of turbulence in the transfer process close to a solid surface. Rather, it might be necessary to give a more detailed account of the eddy motion. In fact if the surface renewal model is used, whereby it is assumed that the fluid in contact with the wall is replaced at fixed time intervals T_c by fluid of bulk concentration, the average rate of mass transfer is given as

$$\overline{K}_\infty = 2(\nu/T_c N_{Sc}\pi)^{1/2} \tag{28}$$

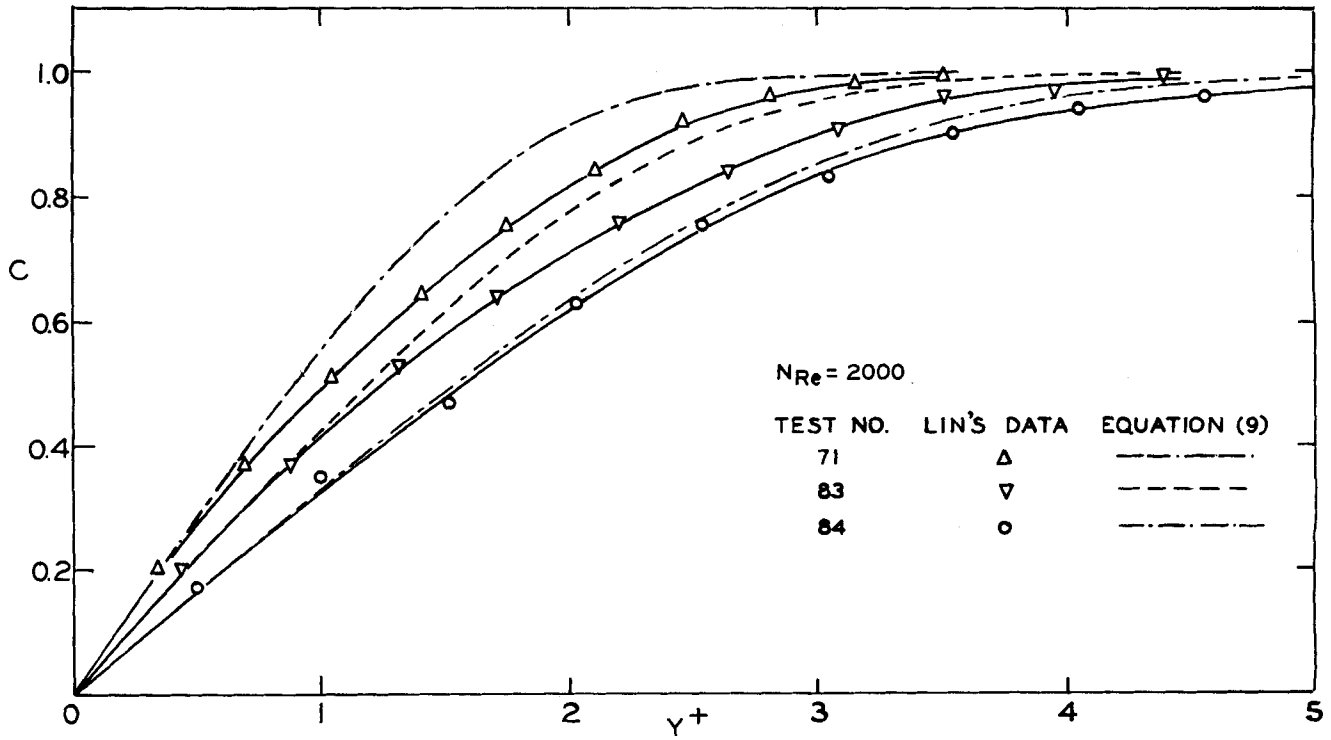

Fig. 9. Concentration profiles in laminar flow.

If T_c is estimated as the reciprocal of the frequency below which is contained half of the energy of the fluctuating signal $<n^+>$, then

$$K_\infty^+ = \left(\frac{4 <n^+>}{\pi N_{Sc}} \right)^{1/2} \quad (29)$$

From the spectra published by Shaw and Hanratty, $<n^+> = 3.1 \times 10^{-4}$. If this is substituted into Equation (29), one obtains $K_\infty^+ = 4.0 \times 10^{-4}$, which is in good agreement with the measured value of $K_\infty^+ = 3.52 \times 10^{-4}$. This agreement is interesting, but it does not establish the validity of the model as a first-order approximation of the wall transfer process. A convincing theory should be based on well-established results on the details of the flow in the immediate vicinity of the wall. It would therefore appear that at present we do not have a firm theoretical guidance for the selection of a proper limiting relation for ϵ.

CONCLUDING REMARKS

A knowledge of the limiting law governing ϵ for $y^+ \rightarrow 0$ is of interest, not only to understand mass transfer data, but also to help in guiding theoretical work on the turbulent flow field close to a wall. Three types of experiments have been outlined for determining this relation: concentration profiles, effect of the length of the transfer section on the rate of mass transfer, and the effect of the Schmidt number on the rate of mass transfer.

The measurement of concentration profiles is the most sound theoretically and the most sensitive. The relation describing the effect of the length of the transfer section on the rate of mass transfer is not too sensitive to the limiting relation for ϵ. The use of data on the effect of Schmidt number requires the further assumption that the coefficient in Equation (6) is independent of the Schmidt number.

Presently available data on mass transfer in pipes or annuli are not as complete as desired, since they do not include accurately measured concentration profiles close to the wall. An analysis of these data gives strong support for a limiting relation of the form of Equation (5). The support for the more explicit Equation (1) is not as strong as one would like, but it is sufficient to recommend the relation as the best interpretation of presently available results.

Studies in rectangular channels were not considered because the presence of secondary flows in channels might not render them comparable with flows in a pipe or in an annulus. This is particularly true for rectangular channels which do not have a large aspect ratio.

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NOTATION

C = concentration
D = molecular diffusion coefficient
d_i = I.D. of an annulus
d_o = O.D. of an annulus
E = coefficient defined by Equation (24)
f = Fanning friction factor, = $0.079\ (N_{Re})^{-1/4}$ for a smooth pipe
J = mass flux
K = local mass transfer coefficient
K^+ = K/u^*
$<K>$ = average mass transfer coefficient over a section of wall
K_∞ = limiting value of K attained in the fully developed region
L = length of the mass transfer section
L^+ = Lu^*/ν
N_{Re} = Reynolds number
N_{Sc} = Schmidt number
n^+ = dimensionless frequency, $n\nu/u^{*2}$
u = velocity in the x direction
u^+ = u/u^*
u^* = friction velocity, $(\tau_w/\rho)^{1/2}$
u_i^* = friction velocity based on shear stress at inner wall
v = velocity in the y direction
v^+ = v/u^*
V = bulk-averaged fluid velocity
W = mass flow rate
x = distance in the flow direction
x^+ = $x\,u^*/\nu$
y = distance perpendicular to the wall
y^+ = $y\,u^*/\nu$

Greek Letters

ϵ = eddy diffusion coefficient
ν = kinematic viscosity
ρ = density
τ = shear stress
τ_w = shear stress at the wall
τ_i = shear stress at the inner wall of an annulus
τ_o = shear stress at the outer wall of an annulus

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